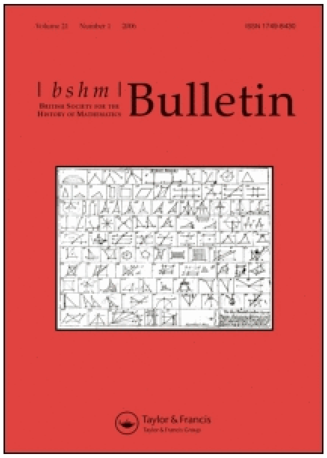

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The tangency problem of Apollonius: three looks

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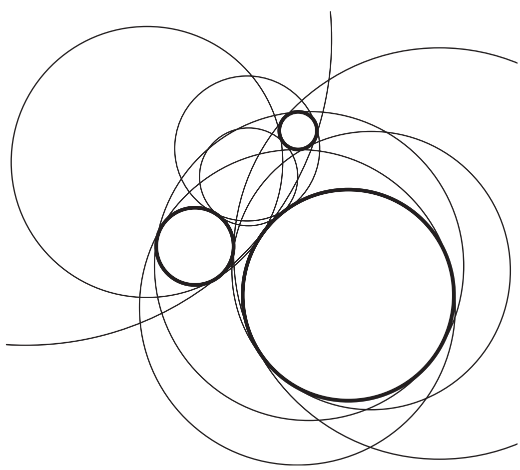
During the great period of Greek geometry, an intriguing construction challenge was posed by Apollonius. Some eighteen centuries later, there was revived interest in the problem, and it continues today. François Viète, Isaac Newton, and Joseph-Diaz Gergonne were among the many mathematicians who solved the problem. They lived in different eras and had very different approaches. Viète stayed close to geometric fundamentals, Newton used his grasp of conics, and Gergonne employed a new understanding of inversion geometry properties.

In the third century BC, after Euclid and after Archimedes, there was Apollonius of Perga, sometimes known as the Great Geometer. Although little is known about the man himself, much of his work has survived in one form or another. Most significantly, the first seven of his eight-volume treatise *Conics* have survived and are still regarded as an authoritative word on the subject.

The tangency problem comes from the two-volume *Tangencies*, which is now lost. The problem is well known, however, and unlike many of the classical Greek geometric construction problems, this one actually has a solution: given three objects, each of them being a circle, a line or a point, construct a circle tangent to all three given objects. Here we must observe a special definition. A point on a circle may be considered tangent to that circle. Since the conditions allow for any combination of circles, lines, and points, there are actually ten construction problems. The simplest is the elementary construction of a circle through three given points. The most challenging case requires a circle tangent to three given circles. That case will receive the most emphasis here.

Five centuries after Apollonius, a copy of *Tangencies* must have been in the hands of Pappus of Alexandria, who wrote the *Synagoge* or *Collection*. The seventh



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book of the *Collection* is called *Treasury of analysis*, and there Pappus discussed the tangency problem. The author stated the problem and introduced some lemmas to support Apollonius's solution, but the solution itself is not there.

It may never have occurred to Pappus that his *Collection* might outlive Apollonius's *Tangencies*, but so it did. It was eventually published in Latin in 1589, and soon there was renewed interest in the tangency problem. François Viète was among the first to solve it. Other solutions followed and the pursuit continues today.

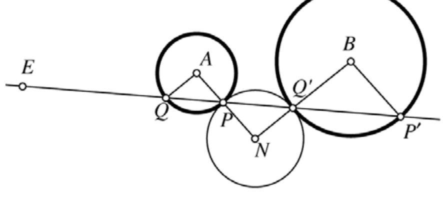
In general, there are eight solutions to the three-circle case, but any or all of them may be impossible depending on the given configuration. As a practical matter, a compass and straight-edge construction of even a single solution can be very difficult, and all eight solutions could overwhelm even the finest draftsman. There are bound to be weak intersections, certain objects will be too large or too small to manage, and there are simply too many marks on the paper. Historically, geometers have contented themselves with describing a construction and proving its validity.

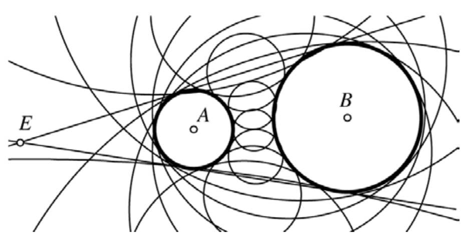
The availability of dynamic geometry software has enabled us to realize better results. We can now plot with pinpoint precision while still adhering to the rules. We can hide construction lines and arcs, bringing them back into view as needed. We can even change the given conditions and have the construction change along with it. In fact, this introduces an added challenge. An ideal solution would fit not only one given set of circles, but any three circles. This gives us the satisfaction of watching the solutions slide into place instantly as we make changes to the given circles. With a clear head we used to visualize such moving images mentally. No doubt Apollonius did too. Now we can actually see them before our eyes.

Two circles

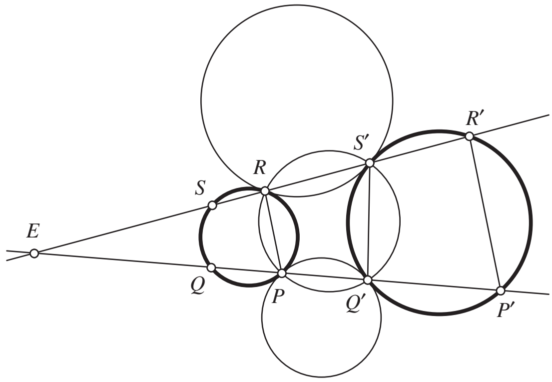
Before working on the three circles, consider a simpler problem beginning with only two given circles. There are infinitely many circles tangent to both. Here is a brief investigation to expose some interesting and useful properties.

Let the given circles have centres A and B, and let E be the external centre of homothety (dilation, similitude) of the two circles. This is the centre of a homothety that maps one circle onto the other. From E draw a secant line through point P on circle A. Let Q be the other point of intersection between line EP and circle A. The line intersects circle B at points P' and Q', the homothetic images of points P and Q. Produce line segments AP and BQ' to their intersection at N. Under the homothety, isosceles triangles PAQ and $P'BQ'$ are similar, so $\angle AQP$, $\angle APQ$, $\angle BQ'P'$ and $\angle BP'Q'$ are all equal. It follows then that $\angle NPQ'$ and $\angle NQ'P$ are also equal, so $NP = NQ'$ and point N is the centre of one of the solution circles.



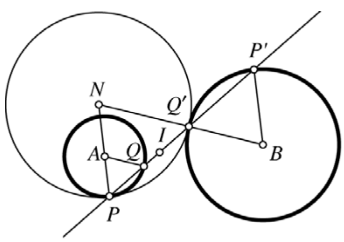


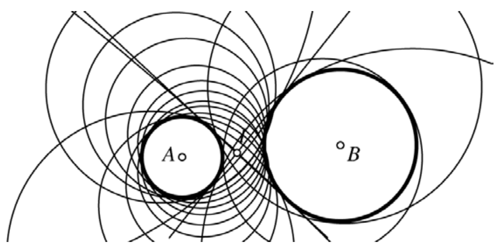
Point P is one of the tangent points and it could have been placed anywhere on circle A, so this method can be used on a whole family of solution circles. All of them are homogenously tangent to the two given circles, which is to say, externally tangent to both or internally tangent to both. Limiting cases in this family are the two common external tangent lines, if they exist.



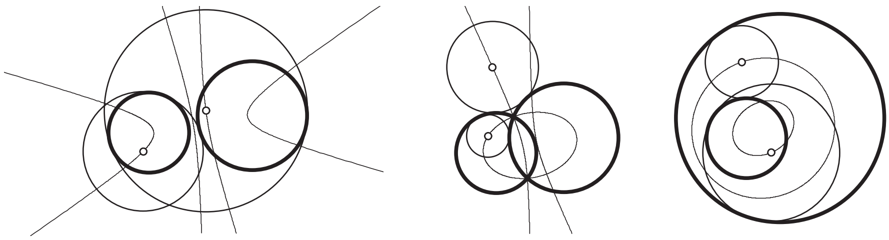
Here is another important property. All of the circles in this family have the same power with respect to point E. Here the same system is shown with two homogeneously tangent circles, one with tangent points P and Q', and the other with R and S'. Angles QPR and $Q'P'R'$ are equal because they are corresponding angles in the homothety. Angle $RS'Q'$ is an exterior angle of cyclic quadrilateral $S'R'P'Q'$ and is therefore equal to the opposite interior angle $Q'P'R'$. Now $RS'Q'P$ is a cyclic quadrilateral because $\angle QPR = \angle RS'Q'$. Therefore, $(ER)(ES') = (EP)(EQ')$, all circles in this tangent family have the same power with respect to point E, and E lies on the radical axis of any two of them.

The same construction works for point I, the internal centre of homothety. Each circle in this family is nonhomogeneously tangent to the two given circles, tangent internally to one and externally to the other. Limiting cases are the two internal common tangent lines, if they exist. All circles in this family have the same power with respect to point I, and I lies on the radical axis of any two of them.





Depending on the relative positions of the given circles, the locus centres for a family of solutions may be a hyperbola or an ellipse. There are also special cases for which the locus may be a line (given congruent circles) or a circle (given concentric circles). In the three-circle problem, picking two pairs from the original three circles, the problem amounts to finding an intersection of two conic sections. This observation has led to some gruelling Cartesian solutions, which, although perfectly correct, are not geometric constructions.

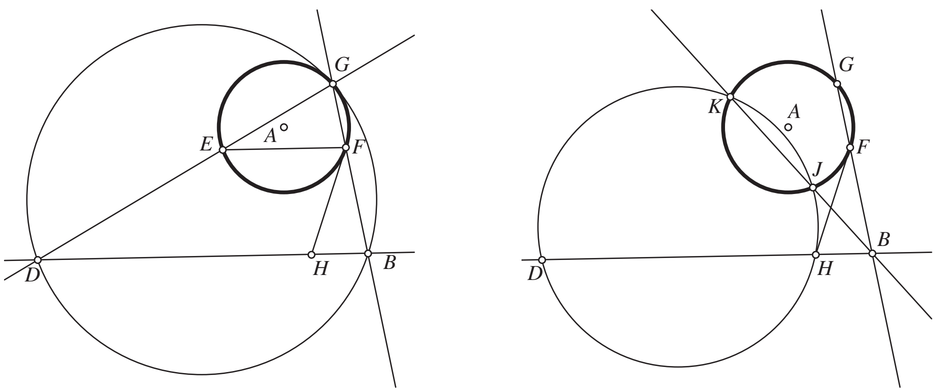


François Viète (1540–1603)

Viète published his solutions in *Apollonius Gallus* (1600). He solved all ten cases in increasing order of complexity, with the later cases making use of solutions in the earlier, simpler cases. To get to the three-circle case, he started with three points and replaced them with circles, one by one. The solutions for the cases involving lines were likewise interdependent.

One circle, two points

The three-point case can be dispensed with, so here we begin with one circle and two points. Let the circle be centred on point A and let the given points be B and D. Suppose the problem is solved. Let G be the point of tangency between the given circle and one of the two solutions. Let GD and GB intersect the given circle A at E and F. The line through F and tangent to circle A intersects DB at H.



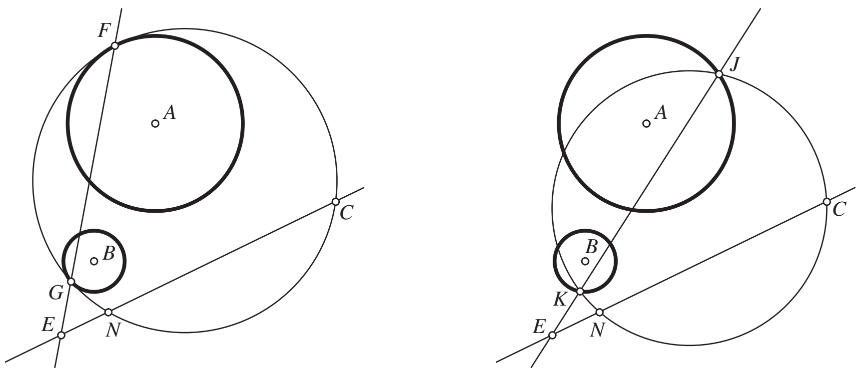
Being their point of tangency, point G is also a centre of homothety of the two circles. Triangles DGB and EGF are similar and DB is parallel to EF. Angles BHF and HFE are equal (alternate interior angles, parallel lines). Angles HFE and EGF are equal (angle in alternate segment). Since angles BHF and EGF are equal, $GFHD$ is a cyclic quadrilateral and $(BH)(BD) = (BF)(BG)$. This is the power of circle A with respect to point B.

Having point H would help bring about the solution. Construct a secant line from B and let it intersect circle A at J and K. Now $(BJ)(BK) = (BF)(BG)$, so $(BH)(BD) = (BJ)(BK)$ and $KJHD$ is a cyclic quadrilateral. Construct circle JKD. It intersects line BD also at H. Construct HF, the tangent segment from H to circle A

(one of two). Line BF intersects circle A also at G, and the circle through B, G and D is a solution. The other tangent segment from H will lead to the other solution.

Two circles, one point

Moving now to the next level, we are given point C and circles with centres A and B. Suppose the problem is solved, and the solution circle is tangent to the given circles at points F and G. As we have seen, points F and G must align with one of the centres of homothety, E, exterior in this example. Let line EC also intersect the solution circle at point N, so $(EN)(EC) = (EG)(EF)$. Let a secant line from point E intersect the given circles at points J and K (one on each circle, but not corresponding points in the homothety centred at E). It has been shown that $(EK)(EJ) = (EG)(EF)$. Therefore, $(EK)(EJ) = (EN)(EC)$ and points K, J, C and N are concyclic.



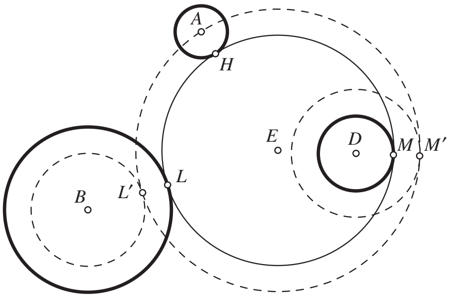
Going back to the start, construct centre of homothety E, secant line EKJ and circle KJC. Line EC intersects this circle also at N. Point N is on a solution circle. Disregard one of the given circles. This is now a case of two points (C and N) and one circle, which was solved above. There are two solutions. Using the other centre of homothety yields two more, for a total of four.

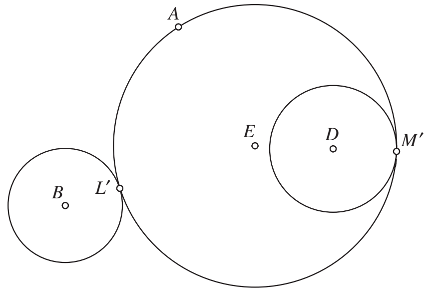
Three circles

The three-circle case generally has eight solutions. To understand how Viète dealt with them, consider only one of them, the circle that is externally tangent to circles A and B, and internally tangent to circle D. Circle A has the smallest radius.

Suppose the problem is solved with point E the centre of the solution circle, which is tangent to the given circles at H, L and M. Translate the tangent points away from point E by distance AH, the radius of the smallest circle. Construct new circles, centred on B and D, and through the respective translation images L' and M'. This effectively reduces the radius of circle B and increases the radius of circle D. Circle A is reduced to a point. There is a circle through A, externally tangent to circle (B, L') and internally tangent to circle (D, M'), having the same centre E. This reduces it to a case of two circles and one point. The other seven solutions are constructed in a similar manner.

There are a lot of things to watch during the construction, particularly the relative sizes of the circles. That is why Viète's solution is not well suited to dynamic geometry software. It works fine for any three given circles, but not in an





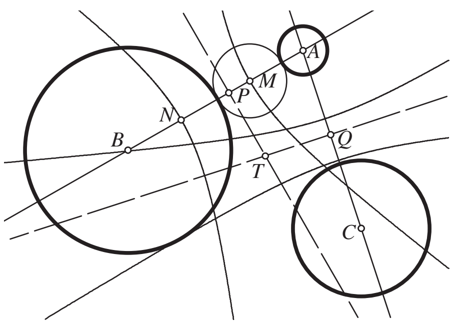
environment where the given conditions can change after completion of the construction.

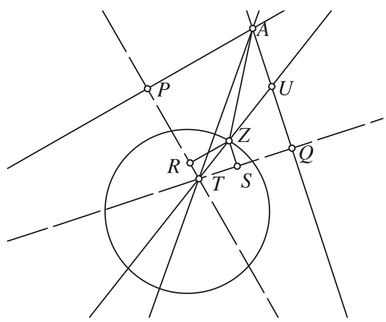
Isaac Newton (1642–1727)

In *Principia mathematica* (1687) Isaac Newton was not specifically addressing the tangency problem, but intentionally or not, he did derive a solution. Lemma XVI from Book I is a construction. Given three points, construct a fourth. The differences of the distances from the constructed point to the three given points are known. This describes an intersection of three hyperbolas. After a very brief description of the solution, Newton acknowledged that the same problem had been solved by Viète. It is clear that Newton was making no effort to restore Apollonius's own solution. Despite his extensive work with conics, Apollonius showed no great interest in the focus or the directrix. Newton's solution depends heavily on those concepts.

The given conditions of Lemma XVI do not include any circles. However, the centres of the three circles and the differences between their radii are equivalent to Newton's conditions. What follows is Newton's construction as applied to the three circles.

Begin with circles centred on points A, B and C. Here the objective will be to construct the centre of the circle that is externally tangent to all three. Construct a circle externally tangent to circles A and B, with its centre, M, on line AB. Point M is a vertex of a hyperbola having foci A and B. Let N be the other vertex with $BN = AM$. The eccentricity of the hyperbola is AB/MN. Construct point P such that A and P divide MN harmonically. Construct a directrix of the hyperbola through P and perpendicular to AB. Using the same procedure, construct a directrix of the hyperbola corresponding to circles A and C, letting Q be the intersection of the axis and the directrix, and let T be the intersection of the two directrices.





Suppose that the problem is solved and that point Z is the centre of the solution circle, an intersection of the two hyperbolas. Points R and S are on PT and QT, and ZR and ZS are perpendicular to the respective directrices. The ratios AZ/ZR and AZ/ZS are the eccentricities of the two hyperbolas, which are both now known. Therefore, ZS/ZR is the ratio of the eccentricities and is constructible. Using this ratio and the directrices, construct the line TZ (one of two). Let this line intersect AQ at U.

Now to find another ratio, $AZ/ZT = (AZ/ZS)(ZS/ZT) = (AZ/ZS)(UQ/UT)$. This ratio is known, as AZ/ZS is the eccentricity of the second hyperbola, and UQ and UT have been constructed. Therefore, AZ/ZT is known. Construct a circle centred on AT, and dividing AT internally and externally in this ratio. This is a circle of Apollonius (bringing us full circle), and is the locus of all points the ratio of whose distances from A and T is AZ/ZT. This circle intersects line TZ at Z. The other intersection is the centre of the internally tangent circle.

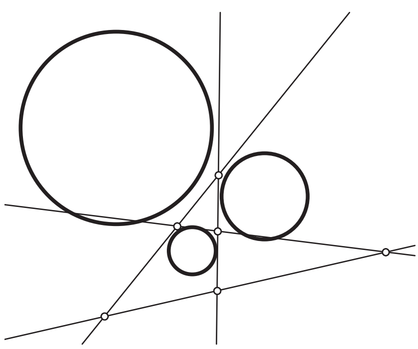
As with Viète, Newton's solution is a bit problematic when it comes to dynamic geometry software. Even after the construction is completed correctly, a change in the size or location of one of the given circles is likely to capsize it. The construction does not simply take care of itself; decisions must be made along the way. This is actually an intersection of two hyperbolas, not three. Two hyperbolas can intersect at four points, only two of which solve the problem.

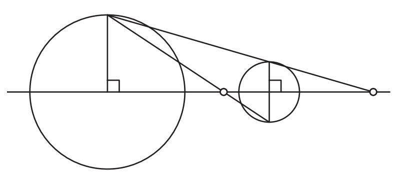
Newton also addressed the simpler, special cases in which two or all three of the distances are equal. It is interesting though that he made no mention of constructing an intersection of two ellipses, or an ellipse and a hyperbola. This same construction applies to those cases as well.

Joseph-Diaz Gergonne (1771–1859)

Apollonius must have had some understanding of inversion geometry, but it would not be until the nineteenth century that the technique became widely appreciated as a way of simplifying construction problems. Joseph-Diaz Gergonne published an inversion-based solution to the tangency problem in his journal, *Annales de Mathématiques*, 4 (1816). The Gergonne solution is especially flexible with regard to the positions of the given circles, and it can also be applied to many of the nine other cases.

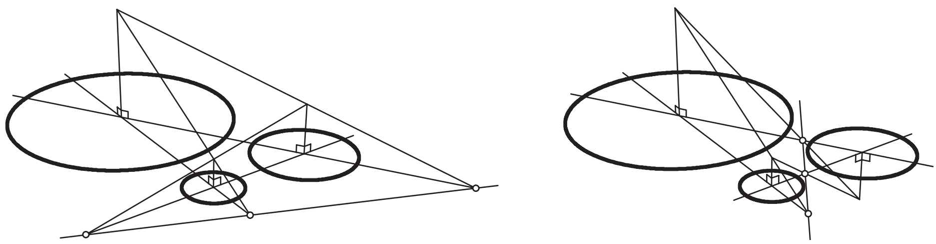
Gergonne's solution makes use of this fact about a system of three circles. Take the circles in three pairs and construct the centres of homothety. The six points fall on four lines, each line including one point of homothety from each pair of circles. Here the lines will be called *lines of similitude*.





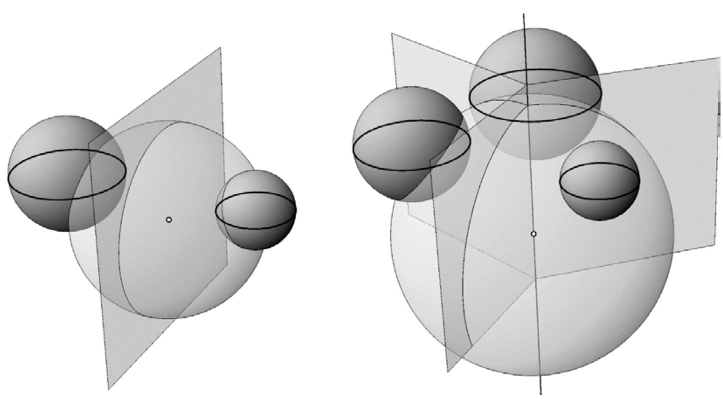
To explain the alignment, use these constructions for the external and internal centres of homothety for two circles. The circles are intersected by radii perpendicular to the axis joining their centres. The line joining the offset points intersects the axis at a centre of homothety, external if the centres are offset on the same side, internal if they are offset on opposite sides.

Now consider three circles in a horizontal plane. An equivalent construction can be effected by leaving the plane. Offset each centre point vertically by a distance equal to the corresponding radius. The plane defined by the three offset points must include three centres of homothety, and it must intersect the plane of the circles in a line. If all three of the circle centres are offset on the same side of the plane, the points constructed are all external centres of homothety. If one centre is offset on the side opposite the others, the result is two internal and one external.

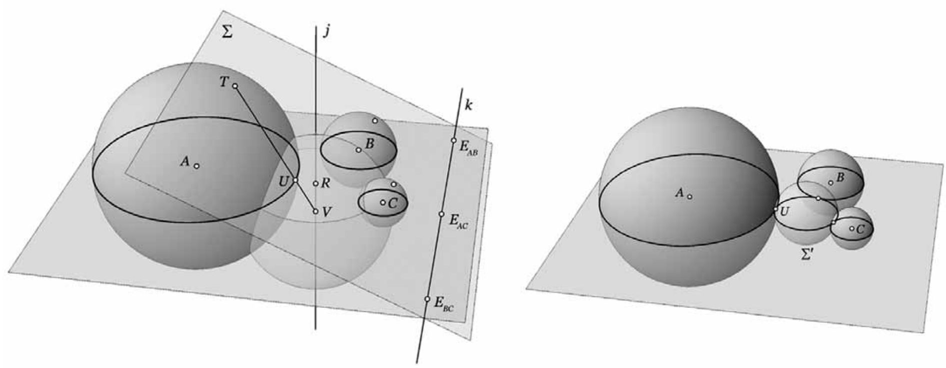


If the reader will forgive a brief diversion, the Gergonne solution will begin with an investigation, which falls short of proof. It is included here as an interesting exercise, an intuitive explanation of the concept. It may even be the thought process that led Gergonne to discover his construction.

Rotate two circles and their radical axis about the axis of symmetry. The circles trace spheres and the radical axis traces a plane. If a sphere has its centre on this plane and is orthogonal to one of the spheres traced by the given circles, then it is orthogonal to both. Now do the same with a system of three circles. The three planes traced by the radical axes intersect at a line perpendicular to the plane of the circles and through the radical centre. If a sphere has its centre on this line and is orthogonal to one of the traced spheres, then it is orthogonal to all three.



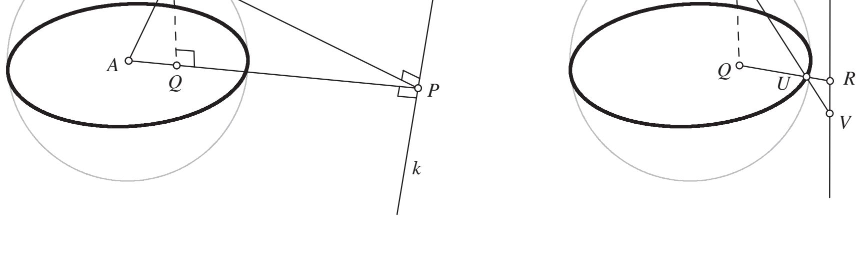
Let A, B, and C be the centres of three circles in a horizontal plane. Let R be the radical centre of the circles and let j be the vertical line through R. Rotate each of the given circles to define a sphere. Lay a plane, Σ, across the top so that it is tangent to all three spheres. Since the plane includes a common tangent line on each pair of spheres, its intersection with the horizontal plane must include the three corresponding centres of homothety. Here that line of similitude is labelled k. Let T be the point of tangency on sphere A. From point T define a line intersecting circle A at U and intersecting line j at V (one of two).



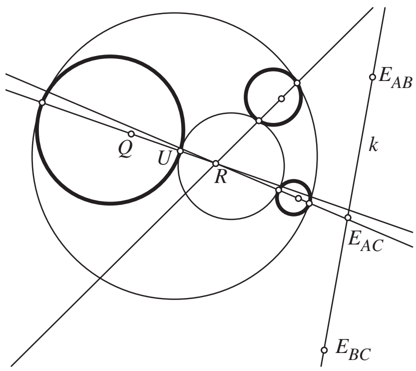
Define a sphere of inversion centred on point V and orthogonal to the other three spheres. The inversion image of plane Σ is sphere Σ', which must be tangent to the images of spheres A, B, and C. Being orthogonal to sphere V, all three of those spheres are invariant under the inversion, so sphere Σ' is tangent to spheres A, B, and C. Point T, being on an invariant sphere, must be mapped to another point on sphere A and on ray VT. That can only be point U, so U is a point of tangency. Since points A and U are in the plane of the circles, so is the centre of sphere Σ'. It follows that the images of the other two tangent points must also be in that plane, and the intersection of the plane and sphere Σ' is one of the solution circles.

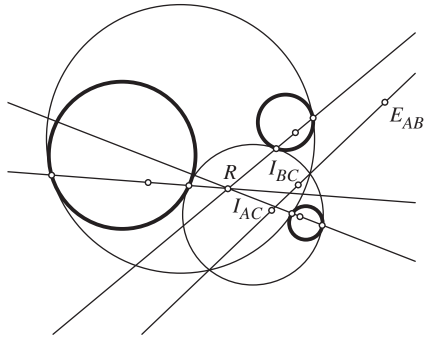
When line TUV was defined, there was a choice of two. The line not chosen would lead to a different solution. Also, plane Σ, rather than being an external common tangent, could have separated one sphere from the other two. Not counting reflections, there are four possible tangent planes, each cutting the horizontal plane at a line of similitude, and each rendering two solutions, eight in all.

Let point P be on line k such that TP is perpendicular to k. Hence, AP is perpendicular to k also. Let Q be the projection of T onto the plane of the circles. Angles ATP and TQP are right angles, so $(AP)(AQ) = (AT)^2$. Since AT is the radius, this makes Q the inverse of P with respect to sphere A or circle A, and Q is the pole of line k with respect to circle A. Points T, U, and V are collinear. Therefore, so are their projections, points Q, U, and R. A similar relationship exists between line k and the other two given circles. The line joining the radical centre, R, and the pole, Q, of a line of similitude in a given circle will intersect that circle at the point of tangency with a solution circle.

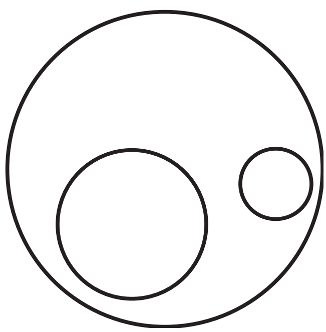


From here, it all comes back down to the plane. Construct the six centres of homothety and the four lines of similitude defined by them. Construct the radical centre. Pick one line of similitude and construct its pole with respect to each of the given circles. Draw a line from the radical centre to each of the poles. These three lines intersect the corresponding circles at points of tangency. Thus each line of similitude brings about two solution circles, eight in all. Two pairs of solutions are shown here.





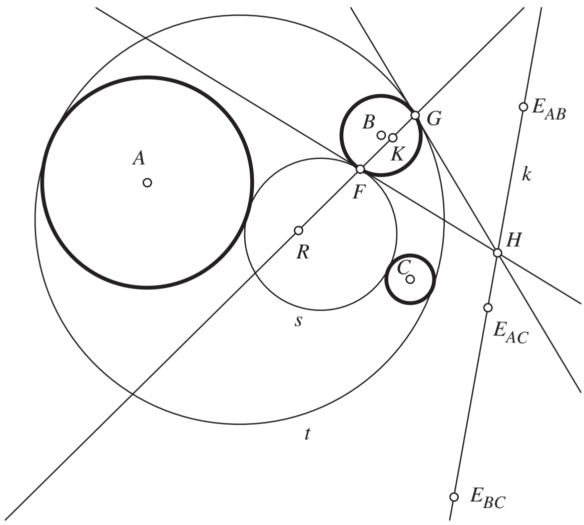
No lies were told here, and the construction is correct, but the proof is not as simple as this. Here is one conflict, certainly not the only one. Consider these three circles. All eight solutions exist, and Gergonne's construction will produce them. The three-dimensional reasoning does not apply here though. When the circles are rolled into spheres, there is no common tangent plane since one sphere encloses the other two.



Consider the solutions in groups, pairs actually. It is possible to define four groups for the solutions without even seeing them. The three given circles can be paired in three different ways. Each solution circle is either homogeneously tangent to exactly one pair (falling into one of three groups) or it is homogeneously tangent to all three circles (the fourth group). Excepting special cases, each group has zero or two solutions. Given the same three circles, let circles s and t be solutions homogeneously tangent to all three.

Now recall the properties that were revealed for the simpler case of two given circles. Since s and t are homogeneously tangent to all three pairs, the three external centres of homothety, E_{AB}, E_{BC} and E_{AC}, all lie on the radical axis of s and t. So k, the line defined by these points, is the radical axis of circles s and t.

If s and t had been the two given circles, then circles A, B, and C would be solutions in the same family, as they are all nonhomogeneously tangent to s and t. Therefore, the internal centre of homothety of s and t must be on the radical axis of circles A and B, and on the radical axis of circles B and C, and on the radical axis of circles A and C. That means that the radical centre, R, of circles A, B, and C is the internal centre of homothety of circles s and t. It also means that the two tangent points on each of the given circles must be collinear with this same point R.



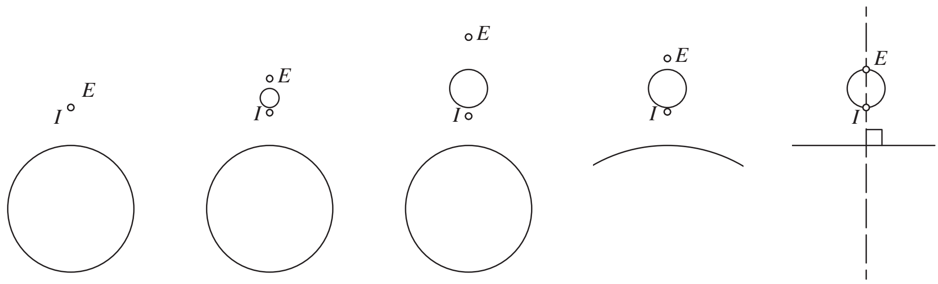
Let F and G be the two tangent points on circle B. Construct lines tangent to circle B at these two points. Point H, the intersection of the tangent lines, is the pole of FG with respect to circle B. Since HF and HG are equal and are the tangent distances from point H to circles s and t, point H must be on line k, the radical axis of s and t. The pole of FG with respect to circle B lies on line k. Therefore, the pole of k lies on FG. Let point K be the pole of line k with respect to circle B. It has been shown here that points R, F, K, and G are collinear. This goes to the heart of the Gergonne solution, and it does so without the use of any tangent planes.

The solutions come in pairs, each pair associated with a line of similitude. This pairing seems to be reoccurring in history with clockwork regularity. Although Newton's approach was entirely different, his construction produced the same pairs. In the early twentieth century, Thomas Heath, working with lemmas of Pappus, derived solutions in the same pairs yet again.

Not surprisingly, Gergonne's construction has certain weaknesses. If the given circles are congruent, the external centres of homothety are sent flying off to points at infinity. Geometry software does not respond well to this, but with some attention, it is not such a problem. All but one of the lines of similitude can be drawn with two internal centres of homothety. The line through the external centres cannot be drawn, but its poles are simply the centres of the given circles.

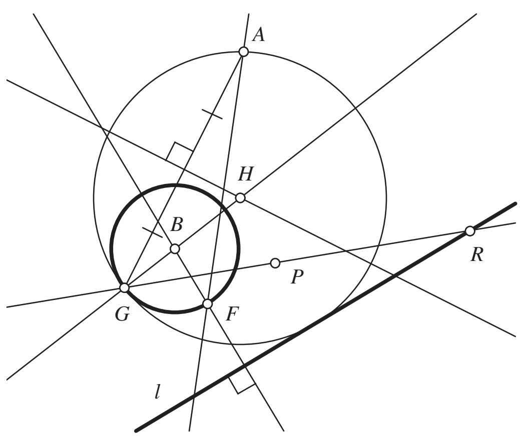
If the given circles are coaxial, the radical centre goes away and so do the poles. This case is not so simple. Newton's construction also fails here, because the directrices do not intersect. Viète's will work.

With some adjustments, the Gergonne construction can be applied to many, but not all, of the other nine cases of the tangency problem. In order to do this, points and lines must be treated as limiting conditions of a circle. Consider the two centres of homothety for two circles. When the radius of one circle is diminished toward zero, that circle and the centres of homothety approach the same point. When one radius approaches infinity, that circle approaches a line, and the centres of homothety approach the points where the axis intersects the other circle.



To see where the construction does not apply, consider what it does in the last few steps. It constructs a point of tangency by intersecting a given circle with a line through a pole with respect to the same circle. In the case of a given point, the intersection could only be the given point itself, which was already known. In the case of a given line, well, there is no pole with respect to a line. The Gergonne construction will only find points of tangency on circles, so in a case having no circles, it is no help at all.

Here is one example of Gergonne's construction used on a simpler case. The given objects are point A, circle B and line l. Point R is the radical centre. A line is constructed through B and perpendicular to l. It intersects the circle at F (the limit of a centre of homothety). Point P is the pole of FA with respect to circle B. Line RP intersects the circle at G, a point of tangency. The centre of a solution circle, point H, is the intersection of BG and the perpendicular bisector of AG. Here, since centre point B was known, it was only necessary to have two points on the solution circle.



With this problem, Apollonius left a challenge which some have found irresistible. Undeterred by precedent, geometers have continued to answer the call, perhaps looking for a better way to solve it, but usually just a different way, their own. There lies the satisfaction.

Note

Paul Kunkel's website has a page with animated constructions of the tangency problem: see http://whistleralley.com/tangents/tangents.htm